Lorentz transformation or Minkowski formula?

Introduction and relevance

In his theory of General Relativity, Einstein replaced the Lorentz transformation by the Minkowski formula. Why did he do that? Obviously, the Lorentz transformation and the Minkowski formula are not the same. The goal of both is the same: calculating the effect of speed in a Euclidean space ("flat" or "straight" space, where Pythagoras is applicable). As a consequence, you can base Einstein's theory of Special Relativity both on the Lorentz transformation (as Einstein did originally), as on the Minkowski formula (as Einstein did later).

We are going to look at the differences and the similarities between both and make understandable why Einstein chose the Minkowski formula as the basis for his theory of General Relativity.

The Lorentz transformation

The Lorentz transformation is given by the following formulas:

t'	$= \gamma.(t - v.x / c^2)$	coordinate t' in S' is function of coordinates t and x in S	(1)
x'	$= \gamma.(x - v.t)$	coordinate x' in S' is function of coordinates t and x in S	(2)
y'	$= y$	coordinate y' in S' equals coordinate y in S	(3)
z'	$= z$	coordinate z' in S' equals coordinate z in S	(4)
γ	$= (1 - v^2 / c^2)^{-½}$	boost-factor "γ"	(5)

In these formulas is S(t,x,y,z) a reference frame in Cartesian coordinates in Euclidean space, and is S'(t',x',y',z') its Lorentz transformed reference frame. The Lorentz transformation ensures the same speed of light "c" in both reference frames. Depending on the separation speed "v" of both reference frames, time dilation in the origin of S' is found by x = v.t:

t'	$= \gamma.(t - v^2.t / c^2)$	coordinate t' in S'(t',0,0,0)	
t'	$= \gamma.(1 - v^2 / c^2).t$	coordinate t' in S'(t',0,0,0)	
t'	$= t / \gamma$	coordinate t' in S'(t',0,0,0)	(6)

Time, as measured on a clock in the origin of S'(t',0,0,0), progresses the boost-factor slower than a clock in a Euclidean reference frame S(t,v.t,0,0). Let us see how that works out according to the Minkowski formula

The Minkowski formula

The differential Minkowski formula as used by Minkowski and Einstein is:

$$ds^2 = c^2.dt_0^2 = c^2.dt^2 - dx^2 - dy^2 - dz^2 \qquad \text{Minkowski space-time (differential)} \qquad (7)$$

Within a Euclidean reference frame S(t,x,y,z), the proper time difference "dt_0" as measured on a local (proper) clock depends on the synchronized clock difference "dt" and the displacements dx, dy, and dz within S(t,x,y,z). This formula can also be written as:

$$ds^2 = c^2.dt_0^2 = c^2.dt^2 - ds^2 \qquad \text{Minkowski space-time (differential)} \qquad (8)$$

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In this formula is $ds^2 = dx^2 + dy^2 + dz^2$, and is "*ds*" the line-element of many solutions to Einstein's theory of General Relativity. Do not confuse space distance "ds" with proper time difference "*ds*" (italic) or "dt_0", since in Einstein's theory of General Relativity c = 1.

So, can the Minkowski formula also guarantee an invariant speed of light? Speed is determined as the change in distance over the change in time. We know from formula (6) that the boost-factor "γ" of light in vacuum is infinite, and thus is the time standing still on the travelling clock: t' = 0. Since "dt_0" is the time difference as measured on this clock, it is zero for light in vacuum. This is confirmed by Einstein. In other words, for light in vacuum with a "dt_0" of zero, the right-hand side of formula (8) leads to:

$ds/dt = c$ speed of light in S(t,x,y,z)

So, the Minkowski formula also guarantees the same and *invariant* speed of light "c" within S(t,x,y,z).
Substitute ds = v.dt in formula (8):

$ds^2 = c^2.dt_0^2 = c^2.dt^2 - v^2.dt^2$ Minkowski space-time (differential)
$ds^2 = c^2.dt_0^2 = (c^2 - v^2).dt^2$
$dt_0^2 = (1 - v^2 / c^2).dt^2$
$dt_0 = dt / \gamma$ time dilation of "dt_0" compared to "dt" (9)

Compare (9) with (6) and the boost-factor "γ" still features in the Minkowski formula. So what is the difference between the Lorentz transformation and the Minkowski formula?

Differences between Lorentz transformation and Minkowski formula

Note that a cube of one cubic meter as measured in S, becomes a different shape between S'(t',0,0,0) and S'(t',1/ γ,1,1). In the Lorentz transformation the length in "x", gets shorter, while, the "y" and "z" coordinate remain the same. Ehrenfest objected to that, how can for example a fast rotating rigid cylinder deform? This is the Ehrenfest paradox.

The Minkowski formula does not have that problem, because the Minkowski formula only relates the synchronized clocks within the reference frame S(t,x,y,z) to the clock and time of a travelling particle with a proper observer, nothing else. In other words, the Lorentz transformation relates two reference frames to each other, while the Minkowski formula is a relation between a travelling clock and one wider Euclidean reference frame with synchronized clocks.

The necessity of synchronized clocks

To apply physics, for example to determine speed, the clocks within a reference frame must be synchronized. When measuring the speed of neutrinos between CERN in Genève and Gran Sasso in Italy, we need to know the distance accurately (ds = 732 km) and the time it took "dt" accurately. The time difference "dt" depends on well synchronized clocks between CERN and Gran Sasso. A problem raised in 2012: the initial thought was that the neutrinos exceeded the speed of light, but that was retracted when the time synchronization turned out to be less than perfect!

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To apply physics within a reference frame, clocks must be synchronized. In order to apply physics in both S(t,x,y,z) and S'(t',x',y',z'), the time "t" has to be the same everywhere in S(t,x,y,z), and the time t' has to be the same everywhere in S'(t',x',y',z'). That is where the Lorentz transformation fails outside of the origin of S'(t',0,0,0). In other words, if the time is synchronized in S(t,x,y,z), the time in S'(t',x',y',z') is not synchronized outside of the origin. The time t' depends on the x-coordinate of S!

The conclusion must be that, within the Lorentz transformed frame, physics may not be applied outside of the origin.

Differential Minkowski vs. absolute Lorentz

Note that the Minkowski formula is a differential formula, the begin situation is irrelevant. This is in contrast to the Lorentz transformation, the transformation is invalidated by a change in speed! In practice, all speeds except the speed of light depend on a change in gravitation. For example, the muons created by the solar wind accelerate towards the earth. This means that the Lorentz transformation is invalidated. The protons circling within the Large Hadron Collider, change direction all the time and are accelerated. This invalidates the Lorentz transformation too.

However, with the Minkowski formula and with Einstein's $E = m.c^2$, we can very accurately describe the lifetime and energy of fast travelling particles.

Conclusion

The Minkowski formula is applicable in any situation and describes the influence of speed of a particle on the proper time difference of the moving proper observer of the particle. Since the Lorentz transformation is not differential, the transformation cannot be used in practice. It is no surprise that Einstein changed to the formula of his tutor Minkowski in Special Relativity and used it as the foundation of his theory of General Relativity!

Want to know more?

The book "Repairing Special Relativity" describes it all in more detail.

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